NONLINEAR STUDIES - www. nonlinearstudies.com

Vol.30, No.2, pp.393-398, 2023

The optimality representation of a finite-life inventory model with exiguous defectives with Hessian matrix approach

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Abstract. Businesses use inventory modelling to estimate the ideal level of inventories to be kept in a production process. This procedure entails controlling the quantity of goods or raw materials to be stored, the frequency of orders, and the supply flow. The primary task would be to reduce the overall cost incurred during the production process. This study examines a finite-life inventory model that permits a very small number of defective products, illustrates the optimal order quantity, and explores how to optimal the second-degree total cost function while validating its optimality with a Hessian matrix. In addition, a numerical analysis is performed, and the findings are described.

1 Introduction

Optimization techniques are often applied in inventory management to minimize the total cost and maximize production. There are several methods in mathematical optimization for linear and non-linear programming problems. To meet its demand, a company should be aware of the optimal order quantity which will reduce its holding and storage costs. In literature, Harris [4] was the first

2010 **Mathematics Subject Classification**: 90B05.

Keywords: Trapezoidal fuzzy numbers, graded-mean, Hessian matrix, Economic order quantity (EOQ), Optimization.

person to study and formulate the economic order quantity (EOQ) model. Fuzzy sets were introduced in the year 1965 by Zadeh [25]. In the year 1983, Zimmermann [26] had proposed fuzzy linear programming to solve linear programming problems by defining the membership functions in fuzzy sets. Park [13] introduced the study of fuzzy sets on inventory models in the year 1987 to obtain EOQ. Hsieh [6] determined two models on economic production quantity in both crisp and fuzzy sense. Salameh [16] studied stochastic models for defective items in a production inventory model. Vujosevic [20] inspected the decision variables on a bounded interval and examined the modification of EOQ. Demand was kept as a constant in classical researches, Donaldson [2] considered a linear demand to attain the optimal order quantity. Wagner [19] minimized the overall cost of an inventory model with setup, holding costs, and demand throughout a period N. Chen [1] studied a backorder inventory model with fuzzifying the order cost, demand, and backorder cost. The first extensive study of economic order quantity resulted in the new aspect of economic production quantity (EPQ) developed by Taft [17]. Zimmermann [27] in his paper proposed many fuzzy applications. Chang [5] examined an inventory model screening the defective items. Wang [20] investigated defective items for an EOQ model by considering it in a fuzzy sense. Yao [21] explored an EPQ model with a fuzzy demand rate. Jing [7] expressed an inventory model and fuzzified the parameters of order quantity and demand. Kalaiarasi et al. [8–10] had compared two different optimization techniques for a non-linear programming problem with sensitivity analysis using trapezoidal fuzzy numbers with the graded-mean method. Roy [15] worked on an inventory model that was dependent on demand with limiting storage capacity. Rosenblatt [12] developed a production process connecting the inputs and outputs of the inventory. Optimizing an inventory model using a geometric programming technique with constrained non-linear programming by using Python for the numerical analysis was done by Kalaiarasi et al. [7]. Stanley et al. [11] developed an ANFIS model to train the datasets of EOQ model and compared with crisp and fuzzy values.

Several researchers applied different optimization techniques for obtaining the optimum values of an inventory model. Velmurugan [19] had applied matrices techniques to study the effectiveness of an inventory model. Hessian matrix optimization technique was applied in problems of queueing models by Emre [3]. Naser [12] had applied the Hessian matrix by forming a second-order partial derivative matrix by keeping the inventory cycle length and backorder quantity as the decision variables. Yang [24] examined an integrated inventory model using matrices to obtain the convexity of the objective function. Kalaiarasi [8] has applied Hessian matrix optimization for inspecting the optimality of the total cost function.

This work intends to minimize the total cost [21] of an inventory model by obtaining the optimal order quantity and optimizing using the Hessian matrix technique for the decision variable. The model was fuzzified by using trapezoidal fuzzy numbers and graded-mean for the defuzzification process.

2 The Hessian matrix optimization technique

The Hessian matrix of a function of n variables $f(y_1, y_2, \ldots, y_n)$ is as follows:

$$\nabla^2 f(y) = \left[\frac{\partial^2 f}{\partial y_i \partial y_j}\right] \quad i, j = 1, 2, \ldots\ldots, n$$

$$H = \begin{bmatrix} \frac{\partial^2 f(y^0)}{\partial y_1^2} & \frac{\partial^2 f(y^0)}{\partial y_1 \partial y_2} \cdots\cdots\cdots & \frac{\partial^2 f(y^0)}{\partial y_1 \partial y_n} \\ \frac{\partial^2 f(y^0)}{\partial y_2 \partial y_1} & \frac{\partial^2 f(y^0)}{\partial y_2^2} \cdots\cdots\cdots\cdots & \frac{\partial^2 f(y^0)}{\partial y_2 \partial y_n} \\ \frac{\partial^2 f(y^0)}{\partial y_n \partial y_1} & \frac{\partial^2 f(y^0)}{\partial y_n \partial y_2} \cdots\cdots\cdots & \frac{\partial^2 f(y^0)}{\partial {y_n}^2} \end{bmatrix}$$

3 The proposed inventory model

S-set-up cost;
D-demand rate at any given time;
P-production rate;
x-defective items in production process;
Q-order quantity from the highest inventory;
H-holding cost of the inventory;
α-a small amount of decay rate (constant);
I-undecayed inventory level.

The total cost function was taken from Uddin [16] having a constant production rate

$$T_C = \frac{S^2 D(P-x-D)}{Q(P-x)} + \frac{Q(H+Ia)^2}{2}. \tag{3.1}$$

Differentiating w.r.t "Q" an equating $\frac{\partial T}{\partial Q} = 0$

$$\frac{\partial T_C}{\partial Q} = -\frac{1}{Q^2}\left[\frac{S^2 D(P-x-D)}{(P-x)}\right] + \frac{(H+Ia)^2}{2}. \tag{3.2}$$

The optimum order quantity is given by

$$Q = \sqrt{\frac{2S^2 D(P-x-D)}{(H+Ia)^2(P-x)}}. \tag{3.3}$$

4 Solution of the crisp EOQ model using the Hessian optimization technique

In view of (3.2)

$$\frac{\partial T_C}{\partial Q} = -\frac{1}{Q^2}\left[\frac{S^2 D(P-x-D)}{(P-x)}\right] + \frac{(H+Ia)^2}{2}$$

$$\frac{\partial^2 T}{\partial Q^2} > 0$$

To find the points of inflection

$$\left(\frac{\partial T_C}{\partial S}, \frac{\partial T_C}{\partial H}\right) = 0$$

The optimal points are given by

$$\left(\frac{2SD(P-x-D)}{Q(P-x)}, \frac{2(H+Ia)}{2}\right) = 0. \tag{4.1}$$

$$|H_{22}| = \begin{vmatrix} \frac{\partial^2 T_C}{\partial S^2} & \frac{\partial^2 T_C}{\partial S \partial H} \\ \frac{\partial^2 T_C}{\partial H \partial S} & \frac{\partial^2 T_C}{\partial H^2} \end{vmatrix}$$

$$|H_{22}| = \begin{vmatrix} \frac{2D(P-x-D)}{Q(P-x)} & 0 \\ 0 & Q \end{vmatrix} > 0$$

$$\therefore \quad |H_{22}| > 0$$

5 Fuzzification using Trapezoidal fuzzy numbers

Using trapezoidal fuzzy numbers for the parameters,

$$\widetilde{T_c} = \frac{S^2\widetilde{D}(P-x-\widetilde{D})}{Q(P-x)} + \frac{Q(H+Ia)^2}{2} \tag{5.1}$$

$$\widetilde{T_c} = \frac{S^2\,(D_1,D_2,D_3,D_4)\,(P-x-(D_1,D_2,D_3,D_4))}{Q(P-x)} + \frac{Q(H+Ia)^2}{2} \tag{5.2}$$

$$\begin{aligned} &= \frac{S^2\,(D_1)\,(P-x-D_1)}{Q(P-x)} + \frac{Q(H+Ia)^2}{2} + \frac{S^2\,(D_2)\,(P-x-(D_2)}{Q(P-x)} + \frac{Q(H+Ia)^2}{2} \\ &+ \frac{S^2\,(D_3)\,(P-x-(D_3)}{Q(P-x)} + \frac{Q(H+Ia)^2}{2} + \frac{S^2\,(D_4)\,(P-x-D_4)}{Q(P-x)} + \frac{Q(H+Ia)^2}{2}. \end{aligned} \tag{5.3}$$

Partially differentiating with respect to 'Q' and solving for Q,

$$\frac{\partial\widetilde{T_c}}{\partial Q} = \frac{Q(H+Ia)^2}{2} - \frac{S^2}{Q^2(P-x)}\,[(D_1)\,(P-x-D_1)+(D_2)\,(P-x-(D_2)+(D_3)\,(P-x-(D_3)+ (D_4)\,(P-x-D_4)] \tag{5.4}$$

$$\Rightarrow \frac{4(H+Ia)^2}{2} - \frac{S^2}{Q^2(P-x)}\,[(D_1)\,(P-x-D_1)+(D_2)\,(P-x-(D_2)+(D_3)\,(P-x-(D_3)+ (D_4)\,(P-x-D_4)] = 0 \tag{5.5}$$

$$\tilde{Q} = \sqrt{\frac{S^2\,[(D_1)\,(P-x-D_1)+(D_2)\,(P-x-D_2)+(D_3)\,(P-x-D_3)+(D_4)\,(P-x-D_4)]}{2(H+Ia)^2(P-x)}}. \tag{5.6}$$

6 Numerical analysis and discussion

The values for the parameters assigned are $S=100, D=2, P=15, x=1, H=2, I=0.01, a=0.02$ and the optimal order quantity is calculated and shown in Table 1. The numerical calculations that take into account backordered items, backorder cost per item with demand, inventory costs for non-defective products, production setup costs with demand, and the rework rate for defective items. The holding cost parameter is fuzzified using trapezoidal fuzzy numbers $\widetilde{H} = (0.5, 1, 3, 3.5)$ and defuzzification done using graded-mean method.

Table 1: The comparison between the crisp and fuzzified optimal values

Parameters	*S = 100 units*	*D= 2 units*	*P=15 units*	*x = 1 units*	*H = 2 units*	*I = 0.01 units*	*a = 0.02 units*
Optimal order value in crisp	*130.924 units*						
Optimal order value before & after fuzzification	***126.414 units***						

7 Conclusion

This paper studies the optimal order quantity of an inventory model that comprises of the total cost function including several parameters. The economic order quantity for the inventory model in crisp and fuzzy sense are derived (refer (3.3) and (5.6)). The minimization of the cost function is validated using Hessian matrix technique (refer eqn.4). Numerical calculations are shown in table 1 that compares both the optimal values in crisp and fuzzy sense. It is seen that the fuzzified value is lesser than the crisp value.

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